Int J Theor Phys (2010) 49: 3209–3215
DOI 10.1007/s10773-009-0213-9

Characterization of Spaces of Filtering States

Mirko Navara

Received: 31 October 2008 / Accepted: 26 November 2009 / Published online: 9 December 2009

Abstract Filtering states on orthomodular lattices have been introduced by G.T. Rüttimann as an "opposite" of completely additive states. He proved that they form a face of the state space. The question *which* faces can be the sets of filtering states remained open. Here we prove that, for any *semi-exposed* face F of a compact convex set C, there is an orthomodular lattice L and an affine homeomorphism φ of C onto the state space of L such that $\varphi(F)$ is the space of filtering states.

Keywords Orthomodular lattice · State · Probability measure · Filtering state · Face

1 Introduction

It has been proved in [16] that any compact convex set can be the state space of an orthomodular lattice. The characterization of the space of σ-additive states has been proved in one direction in [5, 13] and completed by a necessary and sufficient condition in [11]. These spaces were found to be exactly *s-semi-exposed faces* of compact convex sets, i.e., faces which can be expressed as intersections of level sets of some linear functionals.

Inspired by [17], decompositions of states related to complete additivity were studied in [2, 3, 14, 15]. The basic aim was to express any state as a convex combination of a completely additive state and a state which is *far from completely additive*. Here we concentrate on filtering states (introduced in [15]). A state μ is *filtering* if each non-zero element of a lattice dominates a non-zero element of the kernel of μ. A filtering state is *weakly purely finitely additive*, i.e., it cannot be obtained as a non-trivial convex combination of a completely additive state and any other state. (See [3] for an overview of related notions.)

We contribute to the study of the convex structure of the spaces of filtering states. We prove that they can be any semi-exposed faces of compact convex sets. Thus the spaces of filtering states and spaces of σ-additive states have similar characterizations.

Dedicated to the memory of G.T. Rüttimann.

M. Navara (✉)
Center for Machine Perception, Department of Cybernetics, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Praha, Czech Republic
e-mail: navara@cmp.felk.cvut.cz

2 States on Orthomodular Lattices

Here we summarize only some necessary definitions; for more details on orthomodular lattices, we refer to [1, 4, 7, 12].

An *orthomodular lattice* (abbr. OML) is a lattice L with bounds $0, 1$ and with a unary operation $\neg\colon L \to L$ (*orthocomplementation*) such that

$$a \le b \Longrightarrow \neg b \le \neg a,$$

$$\neg\neg a = a,$$

$$a \vee \neg a = 1,$$

$$a \le b \Longrightarrow b = a \vee (\neg a \wedge b) \text{ (\textit{orthomodular law})}.$$

The symmetric relation $a \le b'$ on L is called *orthogonality* and denoted by $a \perp b$. For $a \in L$, the interval $[0, a]_L = \{x \in L \mid x \le a\}$ constitutes, with the ordering and operations naturally inherited from L, an OML (see [12]).

A *measure* on an OML L is an additive mapping $\mu\colon L \to [0, \infty)$, i.e.,

$$\mu(a \vee b) = \mu(a) + \mu(b) \quad \text{whenever } a, b \in L,\ a \perp b.$$

If, moreover

$$\mu\left(\bigvee_{n\in\mathbb{N}} a_n\right) = \sum_{n\in\mathbb{N}} \mu(a_n)$$

for any pairwise orthogonal system $(a_n)_{n\in\mathbb{N}}$ in L for which $\bigvee_{n\in\mathbb{N}} a_n$ exists, then we call μ a σ*-additive measure*. A *state* (= *probability measure*) on an OML L is a measure $\mu\colon L \to [0, 1]$ such that $\mu(1) = 1$. We denote by $\Omega(L)$, resp. $\Omega_\sigma(L)$, the set of all states, resp. all σ-additive states, on L. We consider the state spaces with the product topology. This way, $\Omega(L)$ becomes a compact convex set [13, 14]. An element x of an OML L determines its associated *evaluation functional*

$$\mathbf{e}(x)\colon \Omega(L) \to [0, 1],\ \mu \mapsto \mu(x).$$

An *atom* of a bounded lattice is an element $a \ne 0$ which covers 0, i.e., $\neg\exists b : 0 < b < a$. A lattice L is called *atomic* if for each $b \in L$ there is an atom $a \in L$ such that $a \le b$. If, moreover, each element of L can be expressed as a supremum of atoms, then L is called *atomistic*. A lattice is called *chain-finite* if all its chains (= totally ordered subsets) are finite. A chain-finite OML is atomistic. Here we shall mostly work with atomistic OMLs. We denote by $\mathcal{A}(L)$ the set of all atoms of L. A subset M of a bounded lattice L is called *filtering* if

$$\forall b \in L \setminus \{0\}\ \exists a \in M \setminus \{0\} : a \le b.$$

E.g., the set of all atoms of L is *filtering* iff L is atomic.

Following G.T. Rüttimann [15], a state μ on an OML L is called *filtering* if its kernel, $\mu^{-1}(0)$, is a filtering set, i.e.,

$$\forall b \in L \setminus \{0\}\ \exists a \in L \setminus \{0\} : a \le b,\ \mu(a) = 0.$$

In particular, a necessary condition of a state to be filtering is that it vanishes on all atoms. For atomic OMLs, this condition is also sufficient. We denote by $\Omega_f(L)$ the set of all filtering states on L.

Filtering states play an important role in decompositions of states related to countable and complete additivity, see [2, 3, 5, 14, 15]. These results are generalizations of the Yosida–Hewitt decomposition in Boolean algebras, studied in [17].

3 Descriptions of State Spaces

The study of states on OMLs has been inspired by the following result by R. Greechie:

Lemma 1 [6] *There is a finite OML G admitting no state.*

Remark 1 The proof has been simplified by R. Mayet (personal communication). According to [10], the Mayet's example is optimal with respect to the technique used (although it is not the only example of this size). See also [9] for an overview of related constructions.

Soon after this result by R. Greechie, a complete characterization of state spaces of OMLs has been proved by F. Shultz:

Theorem 1 [16] *The state space of an OML is a convex set, compact in the product topology. Conversely, each compact convex subset of a locally convex Hausdorff topological linear space is affinely homeomorphic to the state space of some chain-finite OML.*

Questions follow how spaces of states with special properties can be described. Often they are found to be *faces* of the state space.

Definition 1 Let C be a compact convex subset of a locally convex Hausdorff topological linear space. A subset F of C is a *face* of C if all $\alpha, \beta, \gamma \in C$, where γ is a convex combination of α, β with nonzero coefficients, satisfy the equivalence

$$\gamma \in F \Longleftrightarrow \alpha, \beta \in F.$$

As an example, for an OML L both $\Omega_\sigma(L)$ and $\Omega_f(L)$ are faces of $\Omega(L)$ [13, 15]. Questions arise which faces can be obtained this way. Answers require the following notions:

Definition 2 [11, 13] Let C be a compact convex set. A face F of C is said to be *exposed* if there exists a continuous affine functional $f: C \to [0,1]$ such that $F = f^{-1}(1) \cap C$. A face of C is said to be *semi-exposed* if it is an intersection of exposed faces. An *s-semi-exposed face* is defined analogously, but the functionals are allowed to be weak* limits of isotone sequences of continuous affine functionals (see [11] for details).

In [11], spaces of *σ-additive states* were characterized as s-semi-exposed faces of compact convex subsets of locally convex Hausdorff topological spaces. Filtering states were introduced by G.T. Rüttimann in [15], their role in decompositions was studied in [2] and [3], but no characterization of the space of filtering states occurred. In the sequel, we partially fill this gap.

4 Preliminary Results on Filtering States

Here we prepare lemmas for the main result that a face which is semi-exposed can be the space of filtering states of some OML. For atomic OMLs, this condition is also necessary:

Lemma 2 *The space of filtering states of an atomic OML L is a semi-exposed face of the state space.*

Proof Due to atomicity, filtering states form the intersection of kernels of all evaluation functionals associated to atoms and

$$\Omega_f(L) = \bigcap_{a\in\mathcal{A}(L)} (\mathbf{e}(a))^{-1}(0) = \bigcap_{a\in\mathcal{A}(L)} (\mathbf{e}(a'))^{-1}(1),$$

where $(\mathbf{e}(a'))^{-1}(1) = \{\mu \in \Omega(L) \mid \mu(a') = 1\}$, $a \in \mathcal{A}(L)$ are exposed faces. □

In the sequel, we shall refer to some constructions with OMLs, in particular the *horizontal sum* (see [4, 7, 12]) and the *substitution of an atom* with an OML (see [8]).

Lemma 3 *There is an OML H which admits exactly one state and this state is filtering.*

Proof Let B be the Boolean algebra of all finite and cofinite subsets of a countable set, say $\mathbb{N}$. The OML H is obtained by the substitution of all atoms (= singletons) in B with copies of the stateless OML G from Lemma 1. In detail, we take the infinite Cartesian product

$$P = \prod_{n\in\mathbb{N}} G$$

and its subsets

$$H_0 = \{(a_1, a_2, \ldots) \in P \mid \{n \in \mathbb{N} \,|a_n \neq 0\} \text{ is finite}\},$$

$$H_1 = \{(a_1, a_2, \ldots) \in P \mid \{n \in \mathbb{N} \,|a_n \neq 1\} \text{ is finite}\}.$$

Notice that $H_1 = \{a' \mid a \in H_0\}$. We take for H the subset $H = H_0 \cup H_1$, which is a subalgebra of P, i.e., it is closed under the lattice operations and orthocomplementation of P. As such, H becomes an OML when equipped with these operations inherited from P.

The OML H is atomistic; its atoms are of the form $a = (a_1, a_2, \ldots)$, where exactly one of the entries a_n, $n \in \mathbb{N}$, is an atom of G and all other entries are zero. Each element of H_0 can be expressed as an orthogonal join of finitely many atoms of H.

Let ν be a state on H. Each atom a of H belongs to a (stateless) factor of P. This factor is contained (as an interval) in H, thus ν vanishes at a, as well as on all finite orthogonal joins of atoms, i.e., on the whole H_0. The value of ν on each element of H_1 must be 1. We proved that H admits only one state. This state vanishes on all atoms. As H is atomistic, the state is filtering. □

Corollary 1 *For each $r \in [0, \infty)$, there is a unique measure μ on the OML H of Lemma 3 such that $\mu(1) = r$.*

Lemma 4 *Let M be an atomic OML, $A \subseteq \mathcal{A}(M)$. Then there is an atomic OML L and an affine homeomorphism $\psi : \Omega(M) \to \Omega(L)$ such that*

$$\Omega_f(L) = \{\psi(\mu) \mid \mu \in \Omega(M),\ A \subseteq \mu^{-1}(0)\}.$$

Proof The OML L is obtained by the substitution of each atom $a \in \mathcal{A}(M) \setminus A$ with a copy of the OML H of Lemma 3. Thus the interval $[0, a]_M$ (isomorphic to the two-element Boolean algebra) is replaced by an interval $[0, a]_L$, isomorphic to H. (We identify the elements corresponding to a in M and L and extend this to the rest of M, considering M a subalgebra of L.) As H and M are atomic, so is also L. Each atom of L is either an element of A or an atom of a copy of H, included during the substitution.

According to Corollary 1, for each state $\mu \in \Omega(L)$ the restriction $\mu \restriction_{[0,a]_L}$ is a measure on $[0, a]_L$ uniquely determined by the value $\mu(a)$. Thus each state $\mu \in \Omega(L)$ is a unique extension of a unique state on M, namely $\mu \restriction_M$. We take for ψ the inverse of this restriction mapping, $\psi^{-1}(\mu) = \mu \restriction_M$. Obviously, ψ is one-to-one, continuous with respect to the product topology, and it preserves affine combinations.

Due to atomicity, a state on L is filtering iff it vanishes on all atoms of L. Any state vanishes on the atoms of copies of H, included during the substitution. Thus a necessary and sufficient condition for $\mu \in \Omega(L)$ to be filtering is

$$\forall a \in A : \mu(a) = 0.$$

□

We shall need the following lemma due to F. Shultz:

Lemma 5 [16, Lemma 3] *Let $p_1, \ldots p_n, q \in \mathbb{R}$. Let L be a chain-finite OML, $x_1, \ldots x_n \in L$. Then L can be embedded (as a subalgebra) into a chain-finite OML M such that a state $\mu \in \Omega(M)$ admits a (unique) extension to M iff*

$$\sum_{i=1}^{n} p_i \mu(x_i) + q = 0.$$

Remark 2 We shall also use the fact that according to the original proof of Lemma 5 atoms of L are atoms of M.

5 Main Theorem

Now we are ready to prove the characterization of spaces of filtering states.

Theorem 2 *Let C be a compact convex set and F a semi-exposed face of C. There is an orthomodular lattice L and an affine homeomorphism $\varphi : C \to \Omega(L)$ such that $\varphi(F) = \Omega_f(L)$ (where $\Omega(L)$, resp. $\Omega_f(L)$, denotes the set of all, resp. all filtering, states on L).*

Proof First, let us solve the case when the face F is exposed. According to Theorem 1, there is a chain-finite OML K and an affine homeomorphism $\chi : C \to \Omega(K)$. The face $\chi(F)$ of $\Omega(K)$ is of the form $\chi(F) = f^{-1}(1) \cap \Omega(K)$ for some continuous affine functional f which maps $\Omega(K)$ into the unit interval $[0, 1]$ of reals. Weak continuity means that f is a finite linear combination of evaluation functionals, i.e., of the form

$$f(\mu) = \sum_{i=1}^{n} p_i \mu(x_i)$$

for some $p_1, \dots p_n \in \mathbb{R}$, $x_1, \dots x_n \in K$.

We take a 4-element Boolean algebra $B = \{0, a, a', 1\}$ and construct the horizontal sum $K \oplus B$. Applying Lemma 5, we embed $K \oplus B$ into an OML M such that any state μ on $K \oplus B$ admits a (unique) extension to M iff μ satisfies the equation

$$\sum_{i=1}^{n} p_i \mu(x_i) + \mu(a) - 1 = 0.$$

Each state μ on K has a unique extension to M, $\mu^* \in \Omega(M)$. It is determined on $(K \oplus B) \setminus K$ by

$$\mu^*(a) = 1 - f(\mu) \in [0, 1], \qquad \mu^*(a') = f(\mu) \in [0, 1],$$

and on $M \setminus (K \oplus B)$ by the uniqueness of the extension. The mapping

$$\eta \colon \Omega(K) \to \Omega(M), \ \mu \mapsto \mu^*,$$

is an affine homeomorphism. The following four conditions are equivalent: (i) $\mu^*(a) = 0$, (ii) $f(\mu) = 1$, (iii) $\mu \in \chi(F)$, (iv) $\mu^* \in \eta(\chi(F))$.

It remains to apply Lemma 4 to OML M and the set $A = \{a\} \subseteq \mathcal{A}(M)$. We get an OML L and an affine homeomorphism $\psi : \Omega(M) \to \Omega(L)$ such that $\psi(\eta(\chi(F))) = \Omega_f(L)$. The mapping $\varphi = \psi \circ \eta \circ \chi : C \to \Omega(L)$ is the required affine homeomorphism.

Second, let us consider a semi-exposed face F. It is an intersection of exposed faces F_α, $\alpha \in I$. We proceed analogously to the first case. We find a chain-finite OML K and an affine homeomorphism $\chi : C \to \Omega(K)$. For each $\alpha \in I$, we take a 4-element Boolean algebra $B_\alpha = \{0, a_\alpha, a'_\alpha, 1\}$, $\alpha \in I$, and we construct the horizontal sum $K \oplus \bigoplus_{\alpha\in I} B_\alpha$. (The sets of atoms of K and B_α, $\alpha \in I$, are disjoint.) Repeated application of Lemma 5 results in an OML M such that each state μ on K has a unique extension to M, $\mu^* \in \Omega(M)$, determined on $(K \oplus \bigoplus_{\alpha\in I} B_\alpha) \setminus K$ by

$$\mu^*(a_\alpha) = 1 - f_\alpha(\mu) \in [0, 1], \qquad \mu^*(a'_\alpha) = f_\alpha(\mu) \in [0, 1],$$

for all $\alpha \in I$. The following three conditions are equivalent: (i) $\forall\alpha \in I : \mu^*(a_\alpha) = 0$, (ii) $\forall\alpha \in I : f_\alpha(\mu) = 1$, (iii) $\mu \in \chi(F)$. We apply Lemma 4 to OML M and the set $A = \{a_\alpha \mid \alpha \in I\} \subseteq \mathcal{A}(M)$. □

The state spaces of finite OMLs are polytopes. All their faces are exposed. Then Theorem 2 reduces to a simpler form:

Corollary 2 *Let C be a polytope and F a face of C. There is an orthomodular lattice L and an affine homeomorphism $\varphi \colon C \to \Omega(L)$ such that $\varphi(F) = \Omega_f(L)$.*

Acknowledgements This research was supported by the Czech Ministry of Education under project MSM 6840770038. The author thanks to the anonymous referee for careful proofreading and remarks which improved the manuscript.

References

1. Beran, L.: Orthomodular Lattices. Algebraic Approach. Academia, D. Reidel, Praha, Dordrecht (1984)
2. De Simone, A., Navara, M.: Yosida–Hewitt and Lebesgue decompositions of states on orthomodular posets. J. Math. Anal. Appl. **255**(1), 74–104 (2001)

3. De Simone, A., Navara, M.: On the Yosida–Hewitt decomposition and Rüttimann decomposition of states. Sci. Math. Jpn. **56**(e6), 49–62 (2002)
4. Dvurečenskij, A., Pulmannová, S.: New Trends in Quantum Structures. Kluwer, Ister, Dordrecht, Bratislava (2000)
5. Fischer, H.R., Rüttimann, G.T.: The geometry of the state space. In: Marlow, A.R. (ed.) Mathematical Foundations of Quantum Theory, pp. 153–176. Academic Press, New York (1978)
6. Greechie, R.J.: Orthomodular lattices admitting no states. J. Combin. Theory A **10**, 119–132 (1971)
7. Kalmbach, G.: Orthomodular Lattices. Academic Press, London (1983)
8. Navara, M.: Constructions of quantum structures. In: Gabbay, D., Lehmann, D., Engesser, K. (eds.) Handbook of Quantum Logic, vol. 1, pp. 335–366. Elsevier, Amsterdam (2007)
9. Navara, M.: Small quantum structures with small state spaces. Int. J. Theor. Phys. **47**(1), 36–43 (2008)
10. Navara, M.: Existence of states on quantum structures. Inf. Sci. **179**, 508–514 (2009). doi:10.1016/j.ins.2008.06.011
11. Navara, M., Rüttimann, G.T.: A characterization of σ-state spaces of orthomodular lattices. Expo. Math. **9**, 275–284 (1991)
12. Pták, P., Pulmannová, S.: Orthomodular Structures as Quantum Logics. Kluwer, Dordrecht/Boston/London (1991)
13. Rüttimann, G.T.: Facial sets of probability measures. Probab. Math. Stat. **6**, 187–215 (1985)
14. Rüttimann, G.T.: Decomposition of cones of measures. Atti Sem. Mat. Fis. Univ. Modena **38**, 267–279 (1990)
15. Rüttimann, G.T.: Weakly purely finitely additive measures. Can. J. Math. **46**, 872–885 (1994)
16. Shultz, F.W.: A characterization of state spaces of orthomodular lattices. J. Comb. Theory A **17**, 317–328 (1974)
17. Yosida, K., Hewitt, E.: Finitely additive measures. Trans. Am. Math. Soc. **72**, 46–66 (1952)